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Spectral Analysis of Infinite-dimensional Dirac Operators on an Abstract Boson-Fermion Fock Space

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Abstract

A review on spectral analysis of infinite dimensional Dirac type operators on an abstract boson-fermion Fock space is presented.

1 Introduction

For each pair $(\mathcal{H}, \mathcal{K})$ of complex Hilbert spaces, the tensor product Hilbert space

$$\mathcal{F}(\mathcal{H}, \mathcal{K}) := \mathcal{F}_b(\mathcal{H}) \otimes \mathcal{F}_f(\mathcal{K})$$

of the boson Fock space

$$\mathcal{F}_b(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathcal{H} = \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \mid \psi^{(n)} \in \bigotimes_s^n \mathcal{H}, \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \right\}$$

over \mathcal{H} and the fermion Fock space

$$\mathcal{F}_f(\mathcal{K}) := \bigoplus_{p=0}^{\infty} \bigwedge^p \mathcal{K} = \left\{ \phi = \{\phi^{(p)}\}_{p=0}^{\infty} \mid \phi^{(p)} \in \bigwedge^p \mathcal{K}, \sum_{p=0}^{\infty} \|\phi^{(p)}\|^2 < \infty \right\}$$

over \mathcal{K} is defined, where $\bigotimes_s^n \mathcal{H}$ denotes the n -fold symmetric tensor product of \mathcal{H} with $\bigotimes_s^0 \mathcal{H} := \mathbb{C}$, $\bigwedge^p \mathcal{K}$ denotes the p -fold anti-symmetric tensor product of \mathcal{K} with $\bigwedge^0 \mathcal{K} := \mathbb{C}$ and, for a vector Ψ in a Hilbert space, $\|\Psi\|$ denotes the norm of Ψ . We call the Hilbert space $\mathcal{F}(\mathcal{H}, \mathcal{K})$ the abstract boson-fermion Fock space over $(\mathcal{H}, \mathcal{K})$. In a previous paper [2], the author introduced a general class of infinite-dimensional

Dirac operators on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ and clarified general mathematical structures behind some supersymmetric quantum field models giving an abstract unification of them. In particular, a path (functional) integral representation of analytical index of an infinite dimensional Dirac operator was derived, which gives a kind of index theorem. But spectral analysis of the infinite dimensional Dirac operators is still missing. Only partial results are available [10]. In the present paper, we review some aspects of spectral analysis of infinite dimensional Dirac operators.

2 Preliminaries

We first recall basic objects and facts associated with Fock spaces. See [11] for more details.

In general, for a linear operator A from a Hilbert space to a Hilbert space, we denote its domain by $D(A)$.

For each vector $f \in \mathcal{H}$, there is a unique densely defined closed linear operator $a(f)$ on $\mathcal{F}_b(\mathcal{H})$ such that its adjoint $a(f)^*$ takes the following form:

$$D(a(f)^*) = \left\{ \psi \in \mathcal{F}_b(\mathcal{H}) \mid \sum_{n=1}^{\infty} \|\sqrt{n} S_n(f \otimes \psi^{(n-1)})\|^2 < \infty \right\},$$

$$(a(f)^* \psi)^{(0)} = 0, \quad (a(f)^* \psi)^{(n)} = \sqrt{n} S_n(f \otimes \psi^{(n-1)}), \quad n \geq 1, \quad \psi \in D(a(f)^*),$$

where S_n denotes the symmetrization operator (symmetrizer) on the n -fold tensor product $\otimes^n \mathcal{H}$ of \mathcal{H} . The operator $a(f)$ (resp. $a(f)^*$) is called the boson annihilation (resp. creation) operator with test vector f .

There is a distinguished vector

$$\Omega_b := \{1, 0, 0, \dots\} \in \mathcal{F}_b(\mathcal{H}),$$

called the boson Fock vacuum in $\mathcal{F}_b(\mathcal{H})$, which is vanished by the annihilation operator:

$$a(f)\Omega_b = 0, \quad \forall f \in \mathcal{H}.$$

The set $\{a(f), a(f)^* \mid f \in \mathcal{H}\}$ of boson annihilation operators and boson creation operators obeys the canonical commutation relations (CCR) over \mathcal{H} :

$$[a(f), a(g)^*] = \langle f, g \rangle_{\mathcal{H}}, \quad [a(f), a(g)] = 0, \quad f, g \in \mathcal{H}$$

on the bosonic finite particle subspace

$$\mathcal{F}_{b,0}(\mathcal{H}) := \{\psi \in \mathcal{F}_b(\mathcal{H}) \mid \exists n_0 \in \mathbb{N} \text{ s.t. } \psi^{(n)} = 0, \forall n \geq n_0\},$$

where $[X, Y] := XY - YX$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product of \mathcal{H} (linear in the second variable).

In general, for a subset \mathcal{E} of a vector space, $\text{span}(\mathcal{E})$ or $\text{span } \mathcal{E}$ denotes the subspace generated by all the vectors of \mathcal{E} .

It is well known that, for each dense subspace \mathcal{D} of \mathcal{H} , the subspace

$$\mathcal{F}_{\text{b,fin}}(\mathcal{D}) := \text{span}\{\Omega_{\text{b}}, a(f_1)^* \cdots a(f_n)^* \Omega_{\text{b}} | n \in \mathbb{N}, f_j \in \mathcal{D}, j = 1, \dots, n\}$$

is dense in $\mathcal{F}_{\text{b}}(\mathcal{H})$. In fact, one has

$$\mathcal{F}_{\text{b,fin}}(\mathcal{D}) = \hat{\otimes}_{\text{s}}^n \mathcal{D},$$

the algebraic n -fold symmetric tensor product of \mathcal{D} .

We next move on to the fermion Fock space $\mathcal{F}_{\text{f}}(\mathcal{K})$. For each $u \in \mathcal{K}$, there is a unique bounded linear operator $b(u)$ on $\mathcal{F}_{\text{f}}(\mathcal{K})$ such that $b(u)^*$ is given as follows:

$$(b(u)^* \phi)^{(0)} = 0, \quad (b(u)^* \phi)^{(p)} = \sqrt{p} A_p(f \otimes \phi^{(p-1)}), \quad p \geq 1, \quad \phi \in \mathcal{F}_{\text{f}}(\mathcal{K}),$$

where A_p is the anti-symmetrization operator (anti-symmetrizer) on $\otimes^p \mathcal{K}$. The operator $b(u)$ (resp. $b(u)^*$) is called the fermion annihilation (resp. creation) operator with test vector u .

The vector

$$\Omega_{\text{f}} := \{1, 0, 0, \dots\} \in \mathcal{F}_{\text{f}}(\mathcal{K})$$

is called the fermion Fock vacuum in $\mathcal{F}_{\text{f}}(\mathcal{K})$, which is vanished by $b(u)$:

$$b(u)\Omega_{\text{f}} = 0, \quad \forall u \in \mathcal{K}.$$

The set $\{b(u), b(u)^* | u \in \mathcal{K}\}$ obeys the canonical anti-commutation relations (CAR) over \mathcal{K} :

$$\{b(u), b(v)^*\} = \langle u, v \rangle_{\mathcal{K}}, \quad \{b(u), b(v)\} = 0, \quad u, v \in \mathcal{K},$$

where $\{X, Y\} := XY + YX$. It follows that

$$\|b(u)\| = \|u\|, \quad \|b(u)^*\| = \|u\|, \quad b(u)^2 = 0, \quad (b(u)^*)^2 = 0, \quad \forall u \in \mathcal{K},$$

where, for a bounded linear operator T on a Hilbert space, $\|T\|$ denotes the operator norm of T .

For each dense subspace \mathcal{D} of \mathcal{K} , the subspace

$$\mathcal{F}_{\text{f,fin}}(\mathcal{D}) := \text{span}\{\Omega_{\text{f}}, b(u_1)^* \cdots b(u_p)^* \Omega_{\text{f}} \mid p \in \mathbb{N}, u_k \in \mathcal{D}, k = 1, \dots, p\},$$

is dense in $\mathcal{F}_{\text{f}}(\mathcal{K})$.

3 Exterior Differential Operators on the Boson-Fermion Fock Space

For a linear operator L on a Hilbert space, we set

$$C^\infty(L) := \cap_{n=1}^\infty D(L^n),$$

the C^∞ -domain of L . If L is self-adjoint, then $C^\infty(L)$ is dense.

Let A be a densely defined closed linear operator from \mathcal{H} to \mathcal{K} . Then, by von Neumann's theorem, A^*A and AA^* are non-negative self-adjoint operators on \mathcal{H} and \mathcal{K} respectively and hence $C^\infty(A^*A)$ and $C^\infty(AA^*)$ are dense in \mathcal{H} and \mathcal{K} respectively. Therefore the algebraic tensor product

$$\mathcal{D}_A^\infty := \mathcal{F}_{b, \text{fin}}(C^\infty(A^*A)) \hat{\otimes} \mathcal{F}_{f, \text{fin}}(C^\infty(AA^*))$$

is dense in the boson-fermion Fock space $\mathcal{F}(\mathcal{H}, \mathcal{K})$.

Proposition 3.1 *There exists a unique densely defined closed linear operator d_A on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ such that the following (i) and (ii) hold:*

(i) $\mathcal{D}_A^\infty \subset D(d_A)$ and \mathcal{D}_A^∞ is a core of d_A .

(ii) For each vector $\Psi \in \mathcal{D}_A^\infty$ of the form

$$\Psi = a(f_1)^* \cdots a(f_n)^* \Omega_b \otimes b(u_1)^* \cdots b(u_p)^* \Omega_f, \quad n, p \geq 0,$$

where $a(f_1)^* \cdots a(f_n)^* \Omega_b$ (resp. $b(u_1)^* \cdots b(u_p)^* \Omega_f$) with $n = 0$ (resp. $p = 0$) should read Ω_b (resp. Ω_f), d_A acts as

$$d_A \Psi = 0 \quad \text{for } n = 0,$$

$$d_A \Psi = \sum_{j=1}^n a(f_1)^* \cdots \widehat{a(f_j)^*} \cdots a(f_n)^* \Omega_b \otimes b(Af_j)^* b(u_1)^* \cdots b(u_p)^* \Omega_f$$

for $n \geq 1$, where $\widehat{a(f_j)^*}$ indicates the omission of $a(f_j)^*$. In particular, d_A leaves \mathcal{D}_A^∞ invariant.

Moreover, the following (iii)–(v) hold:

(iii) $\mathcal{D}_A^\infty \subset D(d_A^*)$ and $d_A^* \Psi = 0$ for $p = 0$,

$$d_A^* \Psi = \sum_{k=1}^p (-1)^{k-1} a(A^*u_k)^* a(f_1)^* \cdots a(f_n)^* \Omega_b \otimes b(u_1)^* \cdots \widehat{b(u_k)^*} \cdots b(u_p)^* \Omega_f$$

for $p \geq 1$. In particular, d_A^* leaves \mathcal{D}_A^∞ invariant.

(iv) $D(d_A^2) = D(d_A)$ and, for all $\Psi \in D(d_A)$, $d_A^2\Psi = 0$.

(v) Let B be a bounded linear operator from \mathcal{H} to \mathcal{K} with $D(B) = \mathcal{H}$. Then, for all $\Psi \in \mathcal{D}_A^\infty$ and $\alpha, \beta \in \mathbb{C}$,

$$\alpha d_A \Psi + \beta d_B \Psi = d_{\alpha A + \beta B} \Psi.$$

We call the operator d_A the exterior differential operator on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ associated with A .

4 Infinite Dimensional Dirac Operators

The Dirac operator on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ associated with A is defined by

$$Q_A := d_A + d_A^*.$$

Theorem 4.1 *The operator Q_A is self-adjoint and unbounded from above and below.*

The Laplace-Beltrami-de Rham operator on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ associated with A is defined by

$$\Delta_A := d_A^* d_A + d_A d_A^*.$$

Theorem 4.2 $\Delta_A = Q_A^2$.

5 Supersymmetric Structure

Let

$$\begin{aligned} \mathcal{F}_+ &:= \mathcal{F}_b(\mathcal{H}) \otimes (\oplus_{p=0}^\infty \wedge^{2p} \mathcal{K}) \quad (\text{even forms}), \\ \mathcal{F}_- &:= \mathcal{F}_b(\mathcal{H}) \otimes (\oplus_{p=0}^\infty \wedge^{2p+1} \mathcal{K}) \quad (\text{odd forms}). \end{aligned}$$

Then we have the orthogonal decomposition

$$\mathcal{F}(\mathcal{H}, \mathcal{K}) = \mathcal{F}_+ \oplus \mathcal{F}_-.$$

Let $P_\pm : \mathcal{F}(\mathcal{H}, \mathcal{K}) \rightarrow \mathcal{F}_\pm$ be the orthogonal projections. Then the operator

$$\Gamma := P_+ - P_-$$

is unitary, self-adjoint and the grading operator for the above orthogonal decomposition.

Proposition 5.1 (*anti-commutativity*) *Operator equality $Q_A\Gamma = -\Gamma Q_A$ holds.*

Corollary 5.2 (*spectral symmetry*) *The spectrum $\sigma(Q_A)$ of Q_A is reflection symmetric with respect to the origin of \mathbb{R} : $\sigma(Q_A) = \sigma(-Q_A)$.*

The quadruple $\text{SQFT}_A := (\mathcal{F}(\mathcal{H}, \mathcal{K}), Q_A, \Delta_A, \Gamma)$ is a supersymmetric quantum theory in the abstract sense [1], where Q_A is a self-adjoint supercharge, Δ_A is the supersymmetric Hamiltonian and Γ is the state-sign operator. We remark that SQFT_A gives a unification of some supersymmetric free quantum field models [2, 3, 4, 5, 6].

6 Relations with Second Quantization Operators

For each self-adjoint operator S on \mathcal{H} , one can define the bosonic second quantization of S by

$$d\Gamma_b(S) := \oplus_{n=0}^{\infty} d\Gamma_b^{(n)}(S)$$

with

$$d\Gamma_b^{(0)}(S) := 0, \quad d\Gamma_b^{(n)}(S) := \overline{\sum_{j=1}^n I \otimes \cdots \otimes I \otimes \overset{j\text{th}}{S} \otimes I \otimes \cdots \otimes I}, \quad n \geq 1,$$

where, for a closable operator T on a Hilbert space, \overline{T} denotes the closure of T . It follows that $d\Gamma_b(S)$ is self-adjoint. If $S \geq 0$, then $d\Gamma_b(S) \geq 0$. Moreover,

$$0 \in \sigma_p(d\Gamma_b(S)), \quad \Omega_b \in \ker(d\Gamma_b(S)).$$

Similarly, for each self-adjoint operator T on \mathcal{K} , one can define the fermionic second quantization of T by

$$d\Gamma_f(T) := \oplus_{p=0}^{\infty} d\Gamma_f^{(p)}(T)$$

with

$$d\Gamma_f^{(0)}(T) := 0, \quad d\Gamma_f^{(p)}(T) := \overline{\sum_{j=1}^p I \otimes \cdots \otimes I \otimes \overset{j\text{th}}{T} \otimes I \otimes \cdots \otimes I}, \quad p \geq 1.$$

It follows that $d\Gamma_f(T)$ is self-adjoint. If $T \geq 0$, then $d\Gamma_f(T) \geq 0$. Moreover,

$$0 \in \sigma_p(d\Gamma_f(T)), \quad \Omega_f \in \ker(d\Gamma_f(T)).$$

As we have already mentioned, the operator A yields the non-negative self-adjoint operators A^*A and AA^* . Therefore A^*A (resp. AA^*) may be a one-particle Hamiltonian for a boson (resp. fermion). Then the Hamiltonian of a non-interacting system consisting of such bosons and fermions is given by

$$H(A) := d\Gamma_b(A^*A) \otimes I + I \otimes d\Gamma_f(AA^*).$$

It follows that $H(A)$ is a non-negative self-adjoint operator acting in $\mathcal{F}(\mathcal{H}, \mathcal{K})$ and

$$0 \in \sigma_p(H(A)), \quad \Omega_b \otimes \Omega_f \in \ker H(A).$$

Theorem 6.1 $H(A) = \Delta_A$. In particular, $H(A)$ is a supersymmetric Hamiltonian.

7 Spectra of $H(A)$ and Q_A

In what follows, we assume that \mathcal{H} and \mathcal{K} are separable. For a linear operator T from a Hilbert space to a Hilbert space, we set

$$\text{nul } T := \dim \ker T \in \{0\} \cup \mathbb{N} \cup \{+\infty\}.$$

Theorem 7.1

$$\begin{aligned} \sigma(H(A)) &= \{0\} \cup \overline{\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma(A^*A) \setminus \{0\}, j = 1, \dots, n \right\}}, \\ \sigma_p(H(A)) &= \{0\} \cup \overline{\bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^n \lambda_j \mid \lambda_j \in \sigma_p(A^*A) \setminus \{0\}, j = 1, \dots, n \right\}}. \end{aligned}$$

Theorem 7.2 The spectrum $\sigma(Q_A)$ and the point spectrum $\sigma_p(Q_A)$ of Q_A are symmetric with respect to the origin and

$$\begin{aligned} \sigma(Q_A) &= \{0\} \cup \overline{\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma(A^*A) \setminus \{0\}, j = 1, \dots, n \right\}}, \\ \sigma_p(Q_A) &= \{0\} \cup \overline{\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma_p(A^*A) \setminus \{0\}, j = 1, \dots, n \right\}} \end{aligned}$$

with

$$\text{nul}(Q_A - \lambda) = \text{nul}(Q_A + \lambda), \quad \lambda \in \sigma_p(Q_A).$$

8 A Simple Perturbation

In this section, we consider a simple perturbation of Q_A via a perturbation of d_A . Let

$$g \in D(A) \setminus \{0\}, \quad v \in D(A^*) \setminus \{0\}$$

and

$$d(\alpha) := d_A + \alpha a(g) \otimes b(v)^*.$$

with a constant $\alpha \in \mathbb{C}$ being a perturbation parameter. It is easy to see that $d(\alpha)$ is densely defined with $D(d(\alpha)) \supset \mathcal{D}_A^\infty$ and

$$d(\alpha)^2 = 0 \quad \text{on } \mathcal{D}_A^\infty.$$

Moreover, $d(\alpha)^*$ is densely defined with $\mathcal{D}_A^\infty \subset D(d(\alpha)^*)$ and

$$d(\alpha)^* = d_A^* + \alpha^* a(g)^* \otimes b(v) \quad \text{on } \mathcal{D}_A^\infty.$$

Hence $d(\alpha)$ is closable. We denote the closure of $d(\alpha) \upharpoonright \mathcal{D}_A^\infty$ by $\bar{d}(\alpha)$.

Lemma 8.1 *For all $\Psi \in D(\bar{d}(\alpha))$, $\bar{d}(\alpha)\Psi$ is in $D(\bar{d}(\alpha))$ and*

$$\bar{d}(\alpha)^2 \Psi = 0.$$

Using the operator $\bar{d}(\alpha)$, one can define a perturbed Dirac operator:

$$Q(\alpha) := \bar{d}(\alpha) + \bar{d}(\alpha)^*.$$

We note that

$$Q(\alpha) = Q_A + V_{g,v}(\alpha) \quad \text{on } \mathcal{D}_A^\infty$$

with

$$V_{g,v}(\alpha) := \alpha a(g) \otimes b(v)^* + \alpha^* a(g)^* \otimes b(v).$$

8.1 Self-adjointness of $Q(\alpha)$

Let $T_{g,v} : \mathcal{H} \rightarrow \mathcal{H}$ be defined by

$$T_{g,v}f := \langle g, f \rangle v, \quad f \in \mathcal{H}.$$

It is obvious that $T_{g,v}$ is a bounded linear operator (a one-rank operator). Hence

$$A(\alpha) := A + \alpha T_{g,v}$$

is a densely defined closed linear operator with $D(A(\alpha)) = D(A)$.

Remark 8.2 *Perturbations of a linear operator by one-rank or two-rank operators have been studied in various contexts. See, e.g. [12, 13] and references therein.*

Lemma 8.3 *(a key lemma) For all $\alpha \in \mathbb{C}$, the following operator equality holds:*

$$\bar{d}(\alpha) = d_{A(\alpha)}.$$

Theorem 8.4

(i) *For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is self-adjoint and*

$$Q(\alpha) = Q_{A(\alpha)}.$$

(ii) *For all $\alpha \in \mathbb{C}$, $Q(\alpha)$ is essentially self-adjoint on \mathcal{D}_A^∞ .*

(iii) *For all $\alpha \in \mathbb{C}$,*

$$Q(\alpha) = \overline{Q_A + V_{g,v}(\alpha)}.$$

(iv) *The operator Γ leaves $D(Q(\alpha))$ invariant and*

$$\Gamma Q(\alpha) + Q(\alpha)\Gamma = 0 \quad \text{on } D(Q(\alpha)).$$

(v) *For all $\Psi \in \mathcal{D}_A^\infty$, the vector-valued function: $\alpha \mapsto Q(\alpha)\Psi$ is strongly continuous on \mathbb{C} . Moreover, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(Q(\alpha) - z)^{-1}$ is strongly continuous in $\alpha \in \mathbb{C}$.*

8.2 Spectra of $Q(\alpha)$

Theorem 8.5 *For all $\alpha \in \mathbb{C}$, $\sigma(Q(\alpha))$ and $\sigma_p(Q(\alpha))$ are symmetric with respect to the origin and*

$$\begin{aligned} \sigma(Q(\alpha)) &= \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma(A(\alpha)^* A(\alpha)) \setminus \{0\}, j = 1, \dots, n \right\} \right), \\ \sigma_p(Q(\alpha)) &= \{0\} \cup \left(\bigcup_{n=1}^{\infty} \left\{ \pm \sqrt{\sum_{j=1}^n \lambda_j} \mid \lambda_j \in \sigma_p(A(\alpha)^* A(\alpha)) \setminus \{0\}, j = 1, \dots, n \right\} \right) \end{aligned}$$

with

$$\text{nul}(Q(\alpha) - \lambda) = \text{nul}(Q(\alpha) + \lambda), \quad \lambda \in \sigma_p(Q(\alpha)).$$

This theorem shows that the spectrum and the point spectrum of $Q(\alpha)$ are completely determined from those of $A(\alpha)^* A(\alpha) \setminus \{0\}$.

8.3 Identification of the domain of $Q(\alpha)$

Recall that $|A| := (A^*A)^{1/2}$ acting in \mathcal{H} . It follows that A is injective if and only if $|A|$ is injective.

Theorem 8.6 *Suppose that A is injective and $g \in D(|A|^{-1})$. Then, for all $|\alpha| < 1/(\|v\| \| |A|^{-1}g \|)$, $Q(\alpha)$ is self-adjoint with $D(Q(\alpha)) = D(Q_A)$ and*

$$Q(\alpha) = Q_A + V_{g,v}(\alpha).$$

Moreover, $Q(\alpha)$ is essentially self-adjoint on any core for Q_A .

Proof. The essential part of the proof is to show that $V_{g,v}(\alpha)$ is Q_A -bounded with a relative upper bound $|\alpha| \|v\| \| |A|^{-1}g \|$. Then one needs only to apply the Kato-Rellich theorem. For more details, see the proof of [10, Theorem 17]. ■

9 Kernel of $Q(\alpha)$

We now investigate the kernel of $Q(\alpha)$. We need a classification for conditions on $\{A, g, v\}$:

(C.1) A is injective, $v \in D(A^{-1})$ and $\langle g, A^{-1}v \rangle \neq 0$. In this case we introduce a constant

$$\alpha_0 := -\frac{1}{\langle g, A^{-1}v \rangle}. \quad (9.1)$$

(C.2) A^* is injective, $g \in D(A^{*-1})$ and $\langle v, A^{*-1}g \rangle \neq 0$. In this case we introduce a constant

$$\beta_0 := -\frac{1}{\langle A^{*-1}g, v \rangle}.$$

(C.3) (a) A is injective and $v \notin D(A^{-1})$ or (b) A is injective and $v \in D(A^{-1})$ with $\langle g, A^{-1}v \rangle = 0$.

(C.4) (a) A^* is injective and $g \notin D(A^{*-1})$ or (b) A^* is injective $g \in D(A^{*-1})$ with $\langle v, A^{*-1}g \rangle = 0$.

We first consider the kernel of $A(\alpha)$ and $A(\alpha)^*$.

Lemma 9.1

(i) *Suppose that (C.1) holds. Then*

$$\begin{aligned} \ker A(\alpha) &= \{0\}, \quad \alpha \neq \alpha_0, \\ \ker A(\alpha_0) &= \{cA^{-1}v | c \in \mathbb{C}\}. \end{aligned}$$

(ii) Suppose that (C.2) holds. Then

$$\begin{aligned}\ker A(\alpha)^* &= \{0\}, \quad \alpha \neq \beta_0, \\ \ker A(\beta_0)^* &= \{cA^{*-1}g | c \in \mathbb{C}\}.\end{aligned}$$

(iii) Suppose that (C.3) holds. Then, for all $\alpha \in \mathbb{C}$,

$$\ker A(\alpha) = \{0\}.$$

(iv) Suppose that (C.4) holds. Then, for all $\alpha \in \mathbb{C}$,

$$\ker A(\alpha)^* = \{0\}.$$

Theorem 9.2

(i) Assume (C.1). Then

$$\ker Q(\alpha_0) = \oplus_{n,p=0}^{\infty} [(\otimes^n \{zA^{-1}v | z \in \mathbb{C}\}) \otimes \wedge^p(\ker A(\alpha_0)^*)].$$

and hence $\text{nul } Q(\alpha_0) = \infty$.

Moreover, for all $\alpha \neq \alpha_0$,

$$\ker Q(\alpha) = \oplus_{p=0}^{\infty} \mathbb{C} \otimes \wedge^p(\ker A(\alpha)^*).$$

(ii) Assume (C.2). Then

$$\begin{aligned}\ker Q(\beta_0) &= \oplus_{n=0}^{\infty} \{[\otimes_s^n \ker(A(\beta_0))] \otimes [\mathbb{C} \oplus \text{span}(\{A^{*-1}g\})]\}, \\ \ker Q(\alpha) &= \oplus_{n=0}^{\infty} [\otimes_s^n \ker A(\alpha) \otimes \mathbb{C}], \quad \alpha \neq \beta_0.\end{aligned}$$

(iii) Assume (C.3). Then, for all $\alpha \in \mathbb{C}$,

$$\ker Q(\alpha) = \oplus_{p=0}^{\infty} [\mathbb{C} \otimes \wedge^p(\ker(A(\alpha)^*))].$$

(iv) Assume (C.4). Then, for all $\alpha \in \mathbb{C}$,

$$\ker Q(\alpha) = \oplus_{n=0}^{\infty} [\otimes_s^n \ker A(\alpha) \otimes \mathbb{C}].$$

Corollary 9.3

(i) Assume (C.1) and (C.2). Then

$$\begin{aligned}\ker Q(\alpha_0) &= \overline{\text{span} \left(\left\{ a(A^{-1}v)^{*n} \Omega_b \otimes b(A^{*-1}g)^{*j} \Omega_f | n \geq 0, j = 0, 1 \right\} \right)}, \\ \ker Q(\alpha) &= \{c\Omega_b \otimes \Omega_f | c \in \mathbb{C}\}, \quad \alpha \neq \alpha_0.\end{aligned}$$

(ii) Assume (C.1) and (C.4). Then

$$\begin{aligned}\ker Q(\alpha_0) &= \overline{\text{span}(\{a(A^{-1}v)^{*n}\Omega_b \otimes \Omega_f | n \geq 0\})}, \\ \ker Q(\alpha) &= \{c\Omega_b \otimes \Omega_f | c \in \mathbb{C}\}, \quad \alpha \neq \alpha_0.\end{aligned}$$

(iii) Assume (C.2) and (C.3). Then

$$\begin{aligned}\ker Q(\beta_0) &= \text{span}(\{\Omega_b \otimes b(A^{*-1}g)^{*j}\Omega_f | j = 0, 1\}). \\ \ker Q(\alpha) &= \{c\Omega_b \otimes \Omega_f | c \in \mathbb{C}\}, \quad \alpha \neq \beta_0.\end{aligned}$$

(iv) Assume (C.3) and (C.4). Then, for all $\alpha \in \mathbb{C}$,

$$\ker Q(\alpha) = \{c\Omega_b \otimes \Omega_f | c \in \mathbb{C}\}.$$

10 Non-zero Eigenvalues of $Q(\alpha)$

Hypothesis (A)

- (i) $\mathcal{H} = \mathcal{K}$;
- (ii) A is an injective and nonnegative self-adjoint operator;
- (iii) $g = v \in D(A^{-1})$.

Under Hypothesis (A), the constant α_0 defined by (9.1) takes the form

$$\alpha_0 = -\frac{1}{\langle v, A^{-1}v \rangle} < 0.$$

Theorem 10.1 *Let Hypothesis (A) be satisfied and $\alpha < \alpha_0$ (< 0). Then, there exists a unique constant $x_0(\alpha) < 0$ such that $\alpha \langle v, (x_0(\alpha) - A)^{-1}v \rangle = 1$ and, for all $n \in \{0\} \cup \mathbb{N}$,*

$$\pm \sqrt{n}x_0(\alpha) \in \sigma_p(Q(\alpha)).$$

with eigenvectors

$$\begin{aligned}&[Q(\alpha) \pm \sqrt{n}x_0(\alpha)] \{a(\phi_\alpha)^{*n-p}\Omega_b \otimes b(\phi_\alpha)^{*p}\Omega_f\} \\ &\in \ker(Q(\alpha) \mp \sqrt{n}x_0(\alpha)) \quad (n \geq p \geq 0),\end{aligned}$$

where

$$\phi_\alpha := (x_0(\alpha) - A)^{-1}v.$$

Moreover, $x_0(\alpha)$, as a function of $\alpha < \alpha_0$, is strictly monotone increasing on $(-\infty, \alpha_0)$ with $\lim_{\alpha \rightarrow -\infty} x_0(\alpha) = -\infty$ and $\lim_{\alpha \rightarrow \alpha_0} x_0(\alpha) = 0$.

Note that Theorem 10.1 holds even if Q_A has no non-zero eigenvalues. This is an interesting phenomenon. Since the condition $\alpha < \alpha_0 < 0$ implies that $|\alpha| > |\alpha_0|$, the phenomenon may be regarded as a strong coupling effect.

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